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TIKHONOV TYPE REGULARIZATION METHODS: HISTORY AND RECENT PROGRESS

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Abstract. *Tikhonov initiated the research on stable methods for the numerical solution of* inverse *and* ill-posed *problems. The theory of* Tikhonov regularization *developed systematically. Till the eighties there has been a success in a rigorous and rather complete analysis of regularization methods for solving linear ill-posed problems. Around 1989 a regularization theory for* non–linear *inverse problems has been developed. About the same time* total variation regularization *for denoising and deblurring of discontinuous data was developed; here, in contrast to classical Tikhonov regularization,the functional is* not *differentiable. The next step toward generalization of regularization methods is* non-convex regularization. *Such regularization models are motivated from* statistics *and* sampling *theory. In this paper we review the history of Tikhonov type regularization models. We motivate non-convex regularization models from statistical consideration, present some preliminary analysis, and support the results by numerical experiments.*

1 Introduction

Inverse problems have been an emerging field over many years. The importance of this field is due to a wide class of applications such as *medical imaging*, including *computerized tomography* (see e.g. Natterer et al [72, 73]), *thermoacoustic imaging* (see e.g. Liu [62], Kruger et al [59]), *electrical impedance tomography* (see e.g. Borcea [8], Cheney, Isaacson & Newell [18], Pidcock [80]). Many of these applications are nowadays assigned to the area of *imaging*. R. West [98] has published the recent survey "In industry seeing is believing", which best documents the importance of this area for industrial applications.

Very frequently with Inverse Problems *ill-posedness* is associated. That is, there are *instabilities* with respect to data perturbations and instabilities in the numerical solution (see e.g. Engl & Hanke & Neubauer [34]). Tikhonov initiated the research on stable methods for the numerical solution of inverse problems. Tikhonov's approach consists in formulating the inverse problem as solving the operator equation

$$F(u) = y \ .$$

Then the solution (presumably it exists) is approximated by a minimizer of the penalized functional

$$\|F(u) - y\|^2 + \alpha\|u - u_0\|^2 \quad (\alpha > 0) \ .$$

Here u_0 is an *a–priori* estimate. Nowadays this approach is commonly referred to as *Tikhonov regularization*. In this paper we solely consider penalized minimization problems (motivated from Tikhonov regularization); for other types of regularization methods, such as *iterative* regularization, we refer to the literature [43, 47, 34, 27].

In the early days of regularization mainly linear ill–posed problems (i.e. F is a linear operator) have been solved numerically. The theory of *regularization* methods developed systematically. Until the eighties there has been success in a rigorous and rather complete analysis of regularization methods for linear ill-posed problems. We refer to the books of Nashed [69], Tikhonov & Arsenin [95], Colton & Kress [20, 21], Morozov [66, 67], Groetsch [43, 41, 42], Natterer [72], Engl & Groetsch [33], Banks & Kunisch [4], Kress [57], Louis [63], Kirsch [55], Engl & Kunisch & Neubauer [34], Bertero & Boccacci [7], Hofmann [51], Rieder [84].

In 1989 Engl & Kunisch & Neubauer [35] and Seidman & Vogel [91] developed an analysis of Tikhonov regularization for *non–linear* inverse problems. Here F is a non-linear, differentiable operator.

About the same time Rudin & Osher & Fatemi [86] (see also Rudin & Osher [85]) introduced *total variation regularization* for denoising and deblurring, which consists in minimization of the functional

$$\mathcal{F}_{ROF}(u) := \|F(u) - y\|^2 + \alpha\|Du\| \, ,$$

where $\|Du\|$ is the bounded variation semi-norm (for a definition of the bounded variation semi-norm we refer to Evans & Gariepy [36]). In contrast to classical Tikhonov regularization, the penalization functional is not differentiable. The Rudin-Osher-Fatemi functional is highly successful in restoring discontinuities in *filtering* and *deconvolution* applications.

The next step toward systematic generalization of regularization methods is *non-convex regularization*. Here the general goal is to approximate the solution of the operator equation by a minimizer of the functional $u \mapsto \int g(F(u) - y, u, \nabla u)$. By *non-convex* we mean that the functional g is non-convex with respect to the third component. From the *calculus of variations* it is well-known that even in the case $F = I$ the non-convexity requires to take into account appropriate minimizing concepts, such a Γ-limits and *quasi-convexification* have to be taken into account (see Dacorogna [22] and Dacorogna & Marcellini [23]). Recently *polyconvex* regularization functionals have been studied by Christensen & Johnson [19] for brain imaging and by Droske & Rumpf [28] for image registration. In [88] non-convex regularization models have been developed for filtering.

The outline of this paper is as follows: in the following sections we review Tikhonov type regularization methods for linear and nonlinear ill-posed problems, total variation regularization. Then we introduce non-convex regularization motivated by statistical considerations and present some preliminary analysis. Moreover, some numerical experiments are presented.

2 Tikhonov Regularization for the Solution of Linear Ill–Posed Problems

In this section we review the method of Tikhonov regularization for the solution of *linear* ill–posed operator equations

$$Lu = y\,, \tag{1}$$

where $L : U \to Y$ is a linear operator between Hilbert spaces U and Y.

Important examples of linear problems are summarized below:

Denoising: In this case $L = I$ and the goal is to recover y from y^δ, which is the data y corrupted by noise. Denoising is an important *pre*-processing step for many applications, such as *segmentation*. Some survey on this aspect is [89] and an introductory article [48].

Evaluation of unbounded operators, respectively numerical differentiation: The goal is to find an approximation of the derivative of y from noise corrupted data y^δ. For some references we refer to Groetsch [40, 44].

Deconvolution and Deblurring: Here $Lu(x) = \int k(|x - y|)u(y)\,dy$ where k is the *convolution* operator, typically k is a *Gaussian* kernel. The general case of solving $Lu(x) = \int k(x, y)u(y)\,dy$ (where k is the *blurring* operator) is called *deblurring*. For a recent reference we refer to Bertero & Boccacci [7].

Computerized Tomography: Here L is the Radon transformation. See e.g. Natterer [72].

Thermoacoustic Imaging: Here L is the *spherical mean operator* (see e.g. Agranovsky & Quinto [3] and Finch & Patch & Rakesh [37]). For applications to imaging we refer to Liu [62], Kruger et al [59, 58, 60] Xu & Wang [102, 100, 101, 103] and Haltmeier et al. [45].

Tikhonov regularization consists in approximation of the solution of (1) by a minimizer of the functional

$$\mathcal{F}_L(u) := \|Lu - y^\delta\|_Y^2 + \alpha \|u\|_U^2 \ . \tag{2}$$

In the functional y^δ denotes *noisy* measurement data of the exact data y, $\|\cdot\|_Y$ and $\|\cdot\|_U$ denote the norm on the Hilbert spaces Y and U, respectively. Typically U and Y are *Sobolev spaces* on a compact domain $\Omega \subseteq \mathbb{R}^n$. For a definition of Sobolev spaces we refer to Adams [2]. In the following, in order to simplify the notation we omit the subscripts U and Y in the definition of the norms. The actual norms will be obvious from the contents.

Since the functional $\mathcal{F}_L$ is strictly convex, the minimizer of $\mathcal{F}_L$ (denoted by u_α^δ) is unique. It is characterized by the solution of the *optimality equation*

$$L^*(Lu - y^\delta) + \alpha u = 0 \ . \tag{3}$$

Here L^* denotes the adjoint of L. The adjoint varies for different norms on the Hilbert spaces U and Y.

In the following we denote by $u^\dagger$ the *minimum norm solution* of (1), that is the solution which is in the orthogonal complement of the null-space of L, $\mathcal{N}^\perp$.

Typical *stability* results for Tikhonov regularization (see e.g. Groetsch [43]) read as follows:

Theorem 2.1. *Let $u^\dagger \in U$ be the minimum norm solution of* (1)*, and let u_α^δ be the regularized solution. Then for $y^\delta \to y =: y^0$*

$$u_\alpha^\delta \to u_\alpha^0 =: u_\alpha \ .$$

This results states that for a fixed positive parameter α the regularized solution is stable with respect to data perturbations.

Convergence results for Tikhonov regularization use information on the noise level $\delta = \|y^\delta - y\|$:

Theorem 2.2. *Let $u^\dagger \in U$ be the minimum norm solution of* (1)*. Let $y^\delta \to y$ and $\alpha(\delta)$ be chosen such that $\alpha(\delta) \to 0$, $\frac{\delta^2}{\alpha(\delta)} \to 0$ for $\delta \to 0$, then*

$$u_{\alpha(\delta)}^\delta \to u^\dagger \ .$$

The later result shows that with an appropriate choice of the regularization parameter α the Tikhonov regularized solution approximates the exact solution $u^\dagger$.

3 Tikhonov Regularization for Non–Linear Ill–Posed Problems

In 1989 Engl & Kunisch & Neubauer [35] and Seidman & Vogel [91] developed an analysis of Tikhonov regularization for *non–linear* inverse problems. Some of these results are reviewed in this section. We consider the solution of the nonlinear operator equation

$$F(u) = y\,, \tag{4}$$

where $F : U \to Y$ is a *non-linear*, *weakly closed* and *continuous* operator between Hilbert spaces U and Y.

Important examples of non-linear inverse problems are

Electrical Impedance Tomography (EIT), which consists in estimating the electrical conductivity a in

$$\nabla \cdot (a \nabla u) = 0 \text{ in } \Omega$$

from pairs of boundary data and measurements $\left(u_i, \frac{\partial u_i}{\partial n}\right)_{i \in I}$ at $\partial\Omega$ for an appropriate index set. For some reference on EIT we refer e.g. to [56, 5, 80, 81, 9, 61, 18, 92, 14, 8, 13] to name but a few.

Inverse Source Problems: See e.g. Hettlich & Rundell [49].

Inverse Scattering: See e.g. Colton & Kress [20, 21].

Tikhonov regularization consists in approximation of the desired solution (4) by the minimizer of the functional

$$\mathcal{F}_N(u) := \|F(u) - y^\delta\|_Y^2 + \alpha \|u - u_0\|_U^2\,. \tag{5}$$

Formally, the essential difference to the linear case is that in the penalizing functional an *a–priori* guess of the solution is introduced. As we see below, the a–priori guess allows a convergence analysis for the *minimum solution* $u^\dagger$ of (4); that is a solution of (4) that minimizes $\|u - u_0\|_U^2$ under all functions u that solve (4) (presumably there exists a solution).

In contrast to the linear setting, the functional $\mathcal{F}_N$ may *no* longer be convex, and the minimizer of $\mathcal{F}_N$ may *not* be unique. If the operator F is Fréchet-differentiable, a minimizer u_α^δ satisfies the *optimality equation*

$$F'(u)^*(F(u) - y^\delta) + \alpha(u - u_0) = 0\,. \tag{6}$$

Here $F'(u)^*$ denotes the adjoint of the Fréchet-derivative $F'(u)$.

Typical *stability* results for Tikhonov regularization (see e.g. Engl & Hanke & Neubauer [34]) read as follows:

Theorem 3.1. *Assume there exists a minimum norm solution of* (4)*, denoted by* $u^\dagger \in U$*. Let* $\{y_k\}_{k\in\mathbb{N}}$ *be a sequence where* $y_k \to y^\delta$ *and let* u_k *be a minimizer of* $\mathcal{F}_N$ *where* y^δ *is replaced by* y_k*. Then there exists a convergent subsequence of* $\{u_k\}_{k\in\mathbb{N}}$ *and the limit of every convergent subsequence is a minimizer of* $\mathcal{F}_N$*.*

In contrast to the linear case just convergence of a subsequence can be proven. This weakness is due to the fact that the minimizer of the Tikhonov functional may not be unique. This result states that for a fixed positive parameter α the regularized solution is stable with respect to data perturbation.

The *convergence* result stated in Engl & Hanke & Neubauer [34] reads as follows:

Theorem 3.2. *Assume there exists* $u^\dagger \in U$*. Let* $\alpha(\delta)$ *be chosen such that*

$$\alpha(\delta) \to 0 \text{ and } \frac{\delta^2}{\alpha(\delta)} \to 0 \text{ for } \delta \to 0 \,.$$

Let $\{y_k\}_{k\in\mathbb{N}}$ *again be a sequence where* $y_k \to y^\delta$*. Then every sequence* $\{u^{\delta_k}_{\alpha(\delta_k)}\}_{k\in\mathbb{N}}$*, where* $\delta_k \to 0$*, and* $u^{\delta_k}_{\alpha(\delta_k)}$ *is a minimizer of* $\mathcal{F}_N$ *with* y^δ *replaced by* y^k*, has a convergent subsequence, and the limit is a* u_0*-minimum-norm-solution. If the* u_0*-minimum-norm-solution is unique, then*

$$u^\delta_{\alpha(\delta)} \to u^\dagger \,.$$

4 Regularization Methods with Convex Non–differentiable Penalty Term

Rudin & Osher & Fatemi [86] (see also [85]) introduced *total variation regularization* for denoising and deblurring. This method consists in minimization of the functional

$$\mathcal{F}_{ROF}(u) := \frac{1}{2}\|F(u) - y\|^2 + \alpha\|Du\|\,,$$

where $\|Du\|$ is the bounded variation semi-norm on a compact domain $\Omega \subseteq \mathbb{R}^n$, which is defined as follows (see e.g. Evans & Gariepy [36])

$$\|Du\| := sup\left\{\int_\Omega u\nabla\cdot\vec{v} : \vec{v} \in C_0^\infty(\Omega;\mathbb{R}^n)\,, |\vec{v}| \leq 1\right\}\,. \tag{7}$$

Here $|\cdot|$ denotes the Euclidean norm and $\nabla\cdot\vec{v}$ is the divergence of a vector valued function $\vec{v}$. For more background on functions of bounded variation we refer to Evans & Gariepy [36]. Note that for a continuously differentiable function u, $\|Du\| = \int_\Omega |\nabla u|$.

Conceptually this functional differs from classical Tikhonov regularization since the penalization functional is not differentiable. The Rudin-Osher-Fatemi functional is highly successful in restoring discontinuities in *filtering* and *deconvolution* applications. The analysis of total variation regularization is significantly more involved since the penalization functional is not differentiable. We refer to Acar & Vogel [1] for a preliminary analysis of total variation regularization methods.

Over the last 10 years various non-differentiable regularization methods have been developed. Their success in image processing is driven by the fact that they allow for *data selective* filtering.

1. Based on statistical considerations Geman and Yang [38] developed *half-quadratic* regularization for image processing applications (see also [17]).

2. Recently there has been a revival of regularization norms based on total variation regularization, where in the definition the Euclidean norm is replaced by some p-norm (see e.g. [77]).

3. In the statistical literature total variation regularization (in a discrete setting for analyzing one dimensional data) is very frequently associated with the *taut-string algorithm* (see Mammen & Geer [64] and Davies & Kovac [24]).

4. The taut-string idea has been extended to *robust, quantile* and *logistic regression* models [29]. In a functional analytical framework an analysis of these models based on G-norm properties has been given in [78]. For a definition of the G-norm we refer to Meyer [65]. *Robust regression* consists in minimization of the functional

   $$\int_\Omega |F(u) - f| + \alpha \|Du\| \ .$$

   Note that here both the fit to data term and the penalization functional are not differentiable. From the statistic literature it is well–known that *robust regression* is capable of handling outliers efficiently.

5. For ϕ convex, Vese [97] studied regularization models of the form

   $$\int_\Omega (Lu - f)^2 + \alpha \int_\Omega \phi(Du)$$

   for *denoising* and *deblurring* on the space of functions of bounded variation. In this case the functional $\int_\Omega \phi(Du)$ is defined via *Fenchel transform* (see Ekeland & Temam [30] and Temam [94]).

   In the discrete setting an analysis of such regularization method has been given by Nikolova [76].

6. To make classical regularization theory applicable for recovery of discontinuous solutions Neubauer et al. [74] used curve representations of discontinuous functions considered of graphs. The single components of the graph functions are regularized by the H^1-Sobolev norm.

For analyzing 1-dimensional discrete data, Steidl & Weickert [93] (see also [93, 12, 68]) investigated under which conditions *soft Haar wavelet shrinkage*, *total variation regularization*, *total variation diffusion*, and a *dynamical system* are equivalent. It is quite notable that in a discretized setting the solution of the total variation flow equation

$$\frac{\partial u}{\partial t} = \left(\frac{u_x}{|u_x|} \right)_x \tag{8}$$

(where the derivatives are replaced by difference quotients) at time α and the minimizer of the discrete total variation regularization correspond. Note that by *semi-group* theory (see e.g. Brezis [11]) total variation regularization corresponds to performing one implicit time step of (8) with step length α.

4.1 Higher Order Derivatives of Bounded Variation

To our knowledge Chambolle & Lions [15] first studied BV-models with second order derivatives for denoising. Their approach consists in minimization of the functional

$$\mathcal{F}_{C-L}(u_1, u_2) := \frac{1}{2} \int_\Omega (u_1 + u_2 - f)^2 + \beta \|Du_1\| + \alpha \|D^2 u_2\| \quad (0 < \alpha, \beta) \ .$$

Here

$$\|D^2 u\| = \int_\Omega |Hu| \, ,$$

where Hu denotes the *Hessian* of u. The asymptotic model, for $\beta \to +\infty$, for *denoising* has been introduced in [87]: the noisy function f is approximated by the minimizer of the functional

$$\mathcal{F}_D(u) := \frac{1}{2} \int_\Omega (u - f)^2 + \alpha \|D^2 u\| \tag{9}$$

over the space of *bounded Hessian* BH. For more background on the space BH we refer to Demengel [25, 26] (see also Evans & Gariepy [36]). The motivation for studying this type of regularization arises from *nondestructive evaluation* to recover discontinuities of a derivative of a potential u in impedance problems. The discontinuities of u are locations of *material defects* (see e.g. Isakov [52, 53]). Later on, second order models for denoising have been considered by Chan & Marquina & Mulet [16]. Moreover, this functional can also be used for recovery of object borders in *low contrast data* (see [50]).

4.2 Other Non-Quadratic Regularization Functionals

Various other non-quadratic regularization models have been developed in statistics (see e.g. Dümbgen & Kovac [29]), where they are commonly referred to as *regression models*. In addition to non-quadratic, non-differentiable regularization functionals there have been proposed a variety non-quadratic, differentiable regularization methods. Some of them have been motivated by applications: Engl & Landl [31, 32] used the convex *maximum entropy* regularization for stabilization. On the other hand driven by the need of efficient numerical methods for solving non-differentiable regularization functionals, differentiable approximations have been derived. See e.g. Chambolle & Lions [15], Nashed & Scherzer [70], Radmoser & Scherzer & Weickert [82, 83, 90], to name but a few.

5 Non-convex Regularization

In this section we present and analyze non-convex regularization models for denoising. Polyconvex regularization models have been used for *image-registration* applications by Christensen & Johnson [19] and Droske & Rumpf [28].

Typically, in image denoising applications, the assumption is that the noise for the intensity values at the single pixels is *uncorrelated* and *Gaussian distributed.* As outlined below, the standard statistical approach of *maximum probability* (MAP) estimator for denoising applications can be considered a quadrature rule of Tikhonov regularization. For a recent survey on the relation between statistics and regularization we refer to Hamza & Krim & Unal [46]. General reference books on statistics and probability theory are [6, 79, 54].

In the following we review the relation between *statistical filtering* and *regularization.* Based on these considerations we derive regularization methods for perturbations in the sampling points. That is, we assume that the persistent noise is due to *sampling* errors.

5.1 Statistical Modelling for Denoising Problems

In the beginning, for the sake of simplicity of presentation, we consider the *one-dimensional sampling problem* to recover a signal u from noisy discrete sample data

$$y_i^\delta = u_i + \delta_i := u(x_i) + \delta_i\,, \quad i = 1, \dots, d\,. \tag{10}$$

That is, we assume that the original signal $u(x_i)$ at the sampling point x_i is perturbed with the noise process δ_i, and therefore the *observed signal* is y_i^δ. A common assumption is that the noise process is *independent* and *identically distributed*, i.e.,

$$\delta = \delta_i\,, \text{ for } i = 1, \dots, d\,.$$

In the sequel we denote by $\vec{y}^\delta = (y_1^\delta, \dots, y_d^\delta)$ the observed signal and by $\vec{u} = (u_1, \dots, u_d)$ the sampled data of the true signal, which is to be estimated.

Let us denote by $p(\vec{u})$ the *prior distribution* of $\vec{u}$, i.e., the probability of the occurrence of $\vec{u}$. Using Bayes theorem [10] we have

$$\log p(\vec{u}|\vec{y}^\delta) + \log p(\vec{y}^\delta) = \log p(\vec{y}^\delta|\vec{u}) + \log p(\vec{u})\,,$$

where $p(\vec{y}^\delta|\vec{u})$ and $p(\vec{u}|\vec{y}^\delta)$ denote the *conditional probabilities.* In particular $p(\vec{u}|\vec{y}^\delta)$ denotes the probability that $\vec{u}$ occurs if $\vec{y}^\delta$ has been observed. Since $\vec{y}^\delta$ is the observed data its probability of occurrence is one and thus

$$\log p(\vec{u}|\vec{y}^\delta) = \log p(\vec{y}^\delta|\vec{u}) + \log p(\vec{u})\,. \tag{11}$$

A *maximum probability* (MAP) estimator $\hat{\vec{u}}$ is characterized to maximize the conditional probability $\log p(\vec{u}|\vec{y}^\delta)$. That is $\hat{\vec{u}}$ is the most likely event if the data $\vec{y}^\delta$ has been observed. If we assume that u_i and δ_i are independent for $i = 1, \ldots, d$, then

$$p(\vec{u}|\vec{y}^\delta) = \prod_{i=1}^{d} p(u_i|y_i^\delta) \text{ and } p(\vec{u}) = \prod_{i=1}^{d} p(u_i) \ .$$

1. If the noise process δ is *Gaussian*, then

   $$p(y_i^\delta|u_i) = K \exp\left(-\frac{\delta^2}{2\sigma^2}\right) = K \exp\left(-\frac{(u_i - y_i^\delta)^2}{2\sigma^2}\right) ,$$

   for $i = 1, \ldots, n$. Here K is a normalizing positive constant and σ^2 denotes the noise variance.

2. A general model for the prior distribution $p(u_i)$ is a *Markov random field* (MRF) [99, 10] which is given by its *Gibbs distribution*

   $$p(u_i) = \frac{1}{Z} \exp\left(-\frac{\Phi(u_i)}{\lambda}\right) .$$

Thus the maximum probability (MAP) estimator $\hat{\vec{u}}$ minimizes the functional

$$\mathcal{F}_S(\vec{u}) := \sum_{i=1}^{d} \left(\Phi(u_i) + \frac{\lambda}{2\sigma^2}(u_i - y_i^\delta)^2\right) .$$

Several models have been proposed in the literature for choosing the prior $\Phi(u)$. A typical choice is

$$\Phi(u_i) := |{u_i}'|^p \text{ with } p \geq 1 ,$$

where ${u_i}'$ is considered an approximation of the gradient of the function with sample data $\vec{u}$. We note that the functional $\mathcal{F}_S$ can be considered a *midpoint quadrature formula* of

$$\int_0^1 \Phi(u(x))\, dx + \frac{\lambda}{2\sigma^2} \int_0^1 (u(x) - y^\delta(x))^2\, dx \ .$$

If $\Phi(u(x)) = |u'(x)|^2$ then this methods is standard *Tikhonov regularization* for denoising (cf. Section 2), and the MAP estimator is the minimizer of the discretized Tikhonov functional.

5.2 Uncertainty in the Sampling Points

The above derivation assumes errors in the observed intensities u_i. In particular it is assumed that the sampling points $\vec{x_i}$ are accurate. In this subsection we derive regularization methods which take into account sampling point errors. That is, we assume that

$$u_i = u(x_i + \delta_i)\,, \quad i = 1, \dots, d\,,$$

where $\delta = \delta_i$ are independent, identically distributed noise processes. Making a Taylor series expansion shows that

$$\frac{u_i - u(x_i)}{u'(x_i)} \approx \delta_i \text{ for } i = 1, \dots, d\,.$$

Following the argumentation of the previous subsection, it can be seen that the MAP estimator $\vec{\hat{u}}$ minimizes the functional

$$\mathcal{F}(\vec{u}) := \sum_{i=1}^{d} \left(\Phi(u_i) + \frac{\lambda}{2\sigma^2} \frac{(u_i - y_i^\delta)^2}{|u_i'|^2} \right)\,. \tag{12}$$

This functional can be considered a quadrature rule for approximating the Tikhonov like functional

$$\mathcal{F}_S(u) := \int_0^1 \Phi(u(x))\,dx + \frac{\lambda}{2\sigma^2} \int_0^1 \frac{(u(x) - y^\delta(x))^2}{|u'(x)|^2}\,dx\,. \tag{13}$$

5.3 Uncertainty in the Level Lines

In this subsection we derive regularization methods for resolving sampling errors for data, ideally defined on uniform regular grid of a square domain in $\mathbb{R}^2$. For arbitrary space dimension the argumentation is analogous. In contrast to the previous section we assume that we have sampling data $\vec{y}^\delta$ of a function u satisfying

$$y_i^\delta = u(\vec{x}_i + \vec{\delta}_i)\,. \tag{14}$$

Here $\vec{\delta} = \vec{\delta}_i$ is a multi-dimensional noise process. We assume that the *level line* $\{u^{-1}(\{u(x_i)\})\}$ can be parameterized and denote by $\vec{\tau}$, $\vec{n}$ the unit tangential, normal direction to the level line, respectively. Then, from Taylor series expansion we find

$$\begin{aligned} y_i^\delta - u(\vec{x}_i) = u(\vec{x}_i + \vec{\delta}_i) - u(\vec{x}_i) &\approx \frac{\partial u}{\partial \vec{\tau}}(x_i) \left\langle \vec{\delta}_i\,, \vec{\tau}(x_i) \right\rangle + \frac{\partial u}{\partial \vec{n}}(x_i) \left\langle \vec{\delta}_i\,, \vec{n}(x_i) \right\rangle \\ &= \frac{\partial u}{\partial \vec{n}}(x_i) \left\langle \vec{\delta}_i\,, \vec{n}(x_i) \right\rangle\,. \end{aligned} \tag{15}$$

The later identity is true since in tangential direction to the level line we have $\frac{\partial u}{\partial \vec{\tau}} = 0$. Let us denote by $\vec{u} := (u_1, \dots, u_n)$ with $u_i := u(\vec{x}_i)$, $i = 1, \dots, d$, then from (15) it follows that

$$\frac{y_i^\delta - u_i}{\frac{\partial u}{\partial \vec{n}}(x_i)} \approx \delta_{\vec{n}}(x_i)\,,$$

where $\delta_{\vec{n}}(x_i) = \vec{\delta}(x_i) \cdot \vec{n}(x_i)$ denotes the noise process in normal direction to the level line, which again is assumed to be independent and Gaussian. For the sake of simplicity of notation we consider $\vec{y}^\delta(x_i)$ the restriction of a function $y^\delta(x_i)$ to the sampling points $\vec{x}_i$, $i = 1, \ldots, d$. We assume that $\frac{y^\delta(x)-u(x)}{\frac{\partial u}{\partial \vec{n}}(x)} \approx \delta_{\vec{n}}(x)$ almost everywhere in Ω. Then, by taking into account that $\left|\frac{\partial u}{\partial \vec{n}}(x)\right| = |\nabla u(x)|$ we find by using the change of variable formula that

$$\int_{p \in \mathbb{R}} \left[\int_{\{u^{-1}(p)\}} \delta_{\vec{n}}^2 d\mathcal{H}^{n-1} \right] dp = \int_{p \in \mathbb{R}} \left[\int_{\{u^{-1}(p)\}} \frac{(u - y^\delta)^2}{|\nabla u|^2} d\mathcal{H}^{n-1} \right] dp = \int_\Omega \frac{(u(x) - y^\delta(x))^2}{|\nabla u(x)|} \, dx \,. \tag{16}$$

Alternatively we could use as a measure of uncertainty

$$\int_\Omega \delta_{\vec{n}}^2(x) \, dx \ = \int_\Omega \frac{(u(x) - y^\delta(x))^2}{|\nabla u(x)|^2} \, dx = \int_{p \in \mathbb{R}} \left[\int_{\{u^{-1}(p)\}} \frac{\delta_{\vec{n}}^2}{|\nabla u|} d\mathcal{H}^{n-1} \right] dp \,. \tag{17}$$

Using a prior $\Phi(u)$, and proceeding as above we end up with minimization of functionals

$$\mathcal{F}_S(u) := \int_\Omega \frac{(u(x) - y^\delta(x))^2}{|\nabla u(x)|^p} \, dx \ + \ \frac{\lambda}{\sigma^2} \int_\Omega \Phi(u)(x) \, dx \text{ with } p = 1, 2 \,. \tag{18}$$

5.4 Existence of a Minimizer: The Case $p = 2$

In the following we restrict our attention to minimization of functional (18), with $p = 2$ and $\Phi(u(x)) = |\nabla u(x)|^2$ on the space $H^1(\Omega)$. For notational convenience we set $\alpha = \frac{\lambda}{2\sigma^2}$ and refer to the according functional $\mathcal{F}_S$ as H^1-functional.

The function

$$f(x, u, \vec{v}) := |\vec{v}|^2 + \alpha \frac{|u - y^\delta(x)|^2}{|\vec{v}|^2} \,.$$

is *non-convex* with respect to $\vec{v}$. It is well-known that minimizers of such functionals have to be considered in a generalized setting (see e.g. Dacorogna [22] Dacorogna & Marcellini [23]). A generalized minimizer is obtained by minimizing the functional

$$\mathcal{F}_S^c(u) := \int_\Omega f_c(x, u(x), \nabla u(x)) \, dx \,, \tag{19}$$

where

$$f_c(x, u, \vec{v}) := \begin{cases} |\vec{v}|^2 + \alpha \frac{|u - y^\delta(x)|^2}{|\vec{v}|^2} & \text{if} \quad |\vec{v}|^2 \geq |u - y^\delta(x)| \alpha^{1/2} \,, \\ 2|u - y^\delta(x)| \alpha^{1/2} & \text{if} \quad |\vec{v}|^2 \leq |u - y^\delta(x)| \alpha^{1/2} \end{cases}$$

is the convex envelope of f. Similar to the proof of Theorem 4.1. in Dacorogna [22] we can deduce the existence of a minimizer $\tilde{u} \in H^1(\Omega)$ of this functional.

5.4.1 Numerical Solution

In this subsection we consider numerical minimization of the functional $\mathcal{F}_S^c$ defined in (19). The derivates of f_c with respect to u and $\vec{v}$ are:

$$\begin{aligned}
D_u f_c(x,u,\vec{v}) &= \begin{cases} 2\,\alpha\frac{u-y^\delta(x)}{|\vec{v}|^2} & \text{if } |\vec{v}|^2 > \sqrt{\alpha}|u-y^\delta(x)| \\ 2\sqrt{\alpha}\frac{u-y^\delta(x)}{|u-y^\delta(x)|} & \text{if } |\vec{v}|^2 \le \sqrt{\alpha}|u-y^\delta(x)| \end{cases} \\
D_{\vec{v}} f_c(x,u,\vec{v}) &= \begin{cases} 2\left(1-\alpha\frac{(u-y^\delta(x))^2}{|\vec{v}|^4}\right)\vec{v} & \text{if } |\vec{v}|^2 > \sqrt{\alpha}|u-y^\delta(x)| \\ 0 & \text{if } |\vec{v}|^2 \le \sqrt{\alpha}|u-y^\delta(x)| \end{cases}
\end{aligned}$$

Thus the minimizer $\hat{u} = \text{argmin } \mathcal{F}_S^c(u)$ solves the optimality condition

$$\begin{array}{rl}
\frac{u(x)-y^\delta(x)}{|\nabla u(x)|^2} - \nabla\left(a(x,u(x),\nabla u(x))\nabla u(x)\right) = 0 & \text{if } |\nabla u(x)|^2 > \sqrt{\alpha}|u(x)-y^\delta(x)| \\
\frac{u(x)-y^\delta(x)}{\sqrt{\alpha}|u(x)-y^\delta(x)|} = 0 & \text{if } |\nabla u(x)|^2 \le \sqrt{\alpha}|u(x)-y^\delta(x)|
\end{array} \tag{20}$$

with

$$a(x,u,\vec{v}) := \frac{1}{\alpha} - \frac{|u-y^\delta(x)|^2}{|\vec{v}|^4}\,.$$

In order to solve (20) we consider the solution of the steady state of the evolution equation:

$$\begin{array}{rl}
\partial_t u - \nabla\left(a(\cdot,u,\nabla u)\nabla u\right) = \frac{y^\delta - u}{|\nabla u|^2} & \text{if } |\nabla u|^2 > \sqrt{\alpha}|u-y^\delta| \\
\partial_t u = \frac{1}{\sqrt{\alpha}}\text{sign}(y^\delta - u) & \text{if } |\nabla u|^2 \le \sqrt{\alpha}|u-y^\delta|
\end{array} \tag{21}$$

For the numerical solution we discretize the equation with finite differences in space and solve the resulting system of ordinary differential equations with an explicit Euler method.

5.5 Existence of a Minimizer: The Case $p = 1$

In the following we restrict our attention to minimization of functional (18) with $p = 1$ and $\Phi(u(x)) = |\nabla u(x)|$. In this case the minimization problem has to be considered on the space of functions of bounded variation. This further complicates the analysis, and to the best of our knowledge no existence results for minimizers are available so far. The numerical results outperform the method for $p = 2$ significantly (see Figure 3 below). The reasons for this is two-fold: First, the investigated data is a piecewise step function and thus of bounded variation, which is further reflected by the total variation regularization term. Moreover, the fit-to-data functional in (18) is motivated from (16), which we think to be more appropriate to the particular data than (17).

For minimizing the functional

$$\mathcal{F}_S(u) := \int_\Omega \frac{(u(x)-y^\delta(x))^2}{|\nabla u(x)|}\,dx + \alpha\int_\Omega |\nabla u(x)|\,dx$$

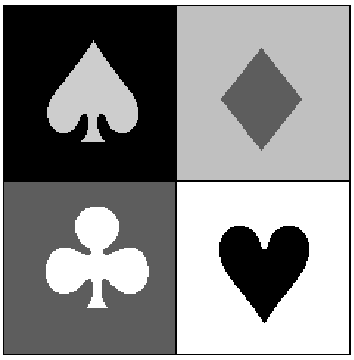

Figure 1: Test image for evaluation of the proposed filters

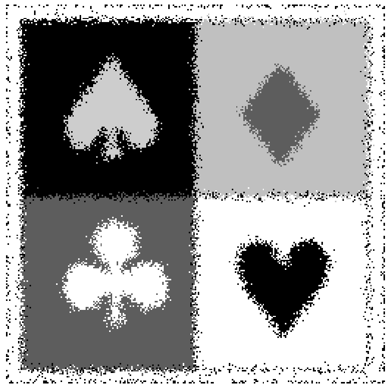

Figure 2: Distorted data obtained by random distortion of the sampling points.

we convexify the function $f(x,u,\vec{v}) := \frac{(u-y^\delta(x))^2}{|\vec{v}|} + \alpha|\vec{v}|$ with to $\vec{v}$. The convexified functional is minimized by solving the (formal) optimality condition (which is again a partial differential equation) with the according evolution process up to a steady state. The evolution process reads as follows:

$$\begin{array}{rl} \partial_t u - \nabla\cdot\left(a_{BV}(\cdot,u,\nabla u)\frac{\nabla u}{|\nabla u|}\right) = \frac{y^\delta - u}{|\nabla u|} & \text{if } |\nabla u| > \sqrt{\alpha}|u-y^\delta| \\ \partial_t u = \frac{1}{\sqrt{\alpha}}\mathrm{sign}(y^\delta - u) & \text{if } |\nabla u| \leq \sqrt{\alpha}|u-y^\delta| \end{array} \tag{22}$$

with

$$a_{BV}(x,u,\vec{v}) := \frac{1}{2}\left(\frac{1}{\alpha} - \frac{|u-y^\delta(x)|^2}{|\vec{v}|^2}\right).$$

5.6 Numerical Results

For evaluating the proposed filter schemes we depict an artificial test image of size 256^2 as shown in Fig. 1. This test image is re-sampled with randomly distorted sampling points (cf. Fig 2). The filtering procedure was performed after scaling the initial grey value to values in $[0,1]$ and assuming pixel size 1.
Figs. 3 and 4 show the result of applying the H^1-Filter and the BV-Filter resp. with $\alpha = .1, \tau = .01$ and 10 iteration steps. Smoothing effects of the filtering schemes can be noticed at the objects edges.

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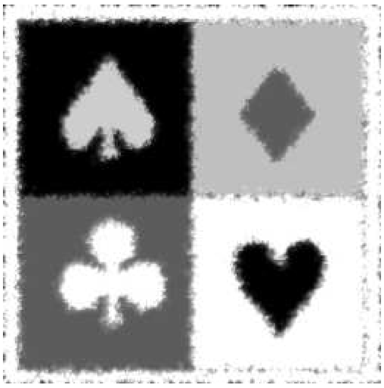

Figure 3: Resulting image after applying the H^1-Filter with $\alpha = .1, \tau = .01$ and 10 iteration steps of the explicit Euler method.

Figure 4: Resulting image after applying the BV-Filter with $\alpha = .1, \tau = .01$ and 10 iteration steps of the explicit scheme.

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